ME 234(b): Optimization Review

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Spring 2022

Slides adapted from textbook: Model Predictive Control (by F. Borrelli, M. Morari, C. Jones)

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An Optimization Problem

Consider a function $f : Z \to \mathbb{R}$ that we want to minimize over a feasible set S. We can write this problem as the following optimization,

$$\begin{aligned} \min_{z} \quad & f(z) \\ \text{s.t} \quad & z \in S \subseteq Z \end{aligned}$$

Our goal is to find the best z^* such that $f(z^*) \leq f(z) \quad \forall z \in S$.

Note that if $S = Z$, then the optimization problem is unconstrained.

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Convexity

A set $S \in \mathbb{R}^s$ is convex if

$$\lambda z_1 + (1-\lambda) z_2 \in S \quad \forall z_1, z_2 \in S,\ \lambda \in [0,1]$$

Similarly, a function $f : S \to R$ is convex iff S is convex and,

$$f(\lambda z_1 + (1-\lambda) z_2) \leq \lambda f(z_1) + (1-\lambda) f(z_2),\ \forall z_1, z_2 \in S,\ \lambda \in [0,1]$$

A function is strictly convex, if it satisfies,

$$f(\lambda z_1 + (1-\lambda) z_2) < \lambda f(z_1) + (1-\lambda) f(z_2),\ \forall z_1, z_2 \in S,\ \lambda \in [0,1]$$

Examples of convex functions:

- A linear function, $f(z) = c'z + r$, is concave and convex.
- A quadratic function $f(z) = z'Hz + 2q'z + r$,
  - Is convex iff $H \succeq 0$ and
  - Is strictly convex if $H \succ 0$.

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Operations that preserve convexity

- Intersection of convex sets is a convex set

$$S = \cap_{i=1} S_i \implies S \text{ is convex when } S_i \text{ are convex}$$

- Sublevel sets are convex

$$f(z) \text{ is convex } \implies S_\alpha = \{z \in S : f(z) \leq \alpha\} \text{ is convex } \forall \alpha \in \mathbb{R}$$

- Nonnegative weighted sum of convex functions is a convex function.

$$f_1, f_2, \ldots, f_N \text{ are convex } \implies f(z) = \sum_{i=1}^{N} \alpha_i f_i(z) \text{ is convex } \forall z \in S, \forall \alpha_i \geq 0$$

- Pointwise maximum of convex functions is a convex function

$$f_1, f_2, \ldots, f_N \text{ are convex } \implies$$
$$f(z) = \max\{f_(z), f_2(z), \ldots, f_n(z)\}, f : S \to \mathbb{R} \text{ is convex}$$

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Convex Optimization Problem

Let's revisit our original optimization problem

$$\min_{z} \quad f(z)$$

$$\text{s.t} \quad z \in S \subseteq Z$$

If f, S are convex, the above optimization is a convex optimization problem.

Theorem: If z^* is the local optimizer for the above convex optimization problem, then z^* is a global optimizer.

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Numerical Methods

In most practical cases, analytically finding an optima is impossible.
So we use numerical methods to get an approximate solution.

The solution of an optimization problem is found in an iterative manner

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Initial guess z^0
Choose $k_{\max}$ such that $|f(z^{k_{\max}}) - f(z^*)| \leq \epsilon$ and $\text{dist}(z^{k_{\max}}, S) \leq \delta$
while $k < k_{\max}$ **do**
    $z^{k+1} = \Psi(z^k, f, S)$
end while

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Stopping criteria

Update Law

Iterate at k+1

Unconstrained OPT: Descent Methods

For unconstrained smooth optimization, we obtain the next iterate from the current iterate by taking a step in a certain direction, i.e.,

$$\min_z f(z), \quad f : \mathbb{R}^s \to \mathbb{R}$$

where, f is convex and continuously differentiable and the update law is

$$z^{k+1} = z^k + h^k d^k$$

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d^k: Direction of descent

h^k: Step size

When $d^k = -\nabla f(z^k)$, the descent method becomes gradient descent. If ∇f is Lipschitz continuous with the constant L, then we can set

$$h^k = \frac{1}{L}$$

Constrained OPT: Gradient Projection

For a constrained, smooth, convex optimization problem,

$$\min_{z} \quad f(z)$$

$$\text{s.t} \quad z \in S \subseteq Z$$

How do we extend the gradient method from the unconstrained setting to the constrained setting?

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Constrained OPT: Gradient Projection

For a constrained, smooth, convex optimization problem,

$$\min_{z} \quad f(z)$$
$$\text{s.t} \quad z \in S \subseteq Z$$

How do we extend the gradient method from the unconstrained setting to the constrained setting?

$$z^{k+1} = \Pi_S\left(z^k - \frac{1}{L}\nabla f(z^k)\right)$$

where,

$$\Pi_S(z) = \arg\min_{y \in S} \frac{1}{2}\|y - z\|^2$$

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Constrained OPT: Gradient Projection

Set Definition	Projection Operator
affine set $S = \{z \in \mathbb{R}^s \mid Az = b\}$ with $(A, b) \in \mathbb{R}^{p \times s} \times \mathbb{R}^p$	$\pi_S(z) = \begin{cases} z + A'(AA')^{-1}(b - Az) & \text{if rank } A = p, \\ z + A'A'^{\dagger}(A^{\dagger}b - z) & \text{otherwise} \end{cases}$
nonnegative orthant $S = \{z \in \mathbb{R}^s \mid z \geq 0\}$	$(\pi_S(z))_i = \begin{cases} z_i & \text{if } z_i \geq 0, \\ 0 & \text{otherwise} \end{cases} \quad i = 1, \ldots, s$
rectangle $S = \{z \in \mathbb{R}^s \mid l \leq z \leq u\}$ with $(l, u) \in \mathbb{R}^s \times \mathbb{R}^s$	$(\pi_S(z))_i = \begin{cases} l_i & \text{if } z_i < l_i, \\ z_i & \text{if } l_i \leq z_i \leq u_i, \\ u_i & \text{if } z_i > u_i \end{cases} \quad i = 1, \ldots, s$
2-norm ball $S = \{z \in \mathbb{R}^s \mid \|z\| \leq r\}, r \geq 0$	$\pi_S(z) = \begin{cases} r\frac{z}{\|z\|} & \text{if } \|z\| > r, \\ z & \text{otherwise} \end{cases}$

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