ME 234(b): MPC with disturbances

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Slides adapted from Model Predictive Control by F. Borrelli, M. Morari, C. Jones

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Recap: MPC Algorithm & its closed–loop properties

$$
\begin{aligned}
J_t^*(x(t)) = \min_{U_t} \quad & x_{t+N|t}^T P_\infty x_{t+N|t} + \sum_{k=t}^{t+N-1} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t} \right) \\
\text{s.t.} \quad & x_{k+1|t} = A x_{k|t} + B u_{k|t}, \quad \forall k \in \{t, t+1, \dots t+N-1\} \quad \text{(OPT)} \\
& x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \quad \forall k \in \{t, t+1, \dots t+N-1\} \\
& x_{t+N|t} \in \mathcal{X}_F
\end{aligned}
$$

while $x_t \neq x_{\text{goal}}$ **do**
  Measure initial state at time t, $x_t = x(t)$
  Solve (OPT) to get the optimal control U_t
  if $U_t \neq \emptyset$ **then**
    Apply the first control input $U_t(1)$
  end if
  Wait for the new sampling time, $t = t + \Delta t$.
end while

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\text{s.t.} \quad & x_{k+1|t} = A x_{k|t} + B u_{k|t}, \quad \forall k \in \{t, t+1, \dots t+N-1\} \quad \text{(OPT)} \\
& x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \quad \forall k \in \{t, t+1, \dots t+N-1\} \\
& x_{t+N|t} \in \mathcal{X}_F
\end{aligned}
$$

(OPT) is **recursively feasible** if the terminal set is control invariant.

(OPT) is **stable** if the terminal weight satisfies the Discrete-time Algebraic Riccati equation (DARE)

$$
P_\infty = A^T P_\infty A + Q - A^T P_\infty B (B^T P_\infty B + R)^{-1} B^T P_\infty A
$$

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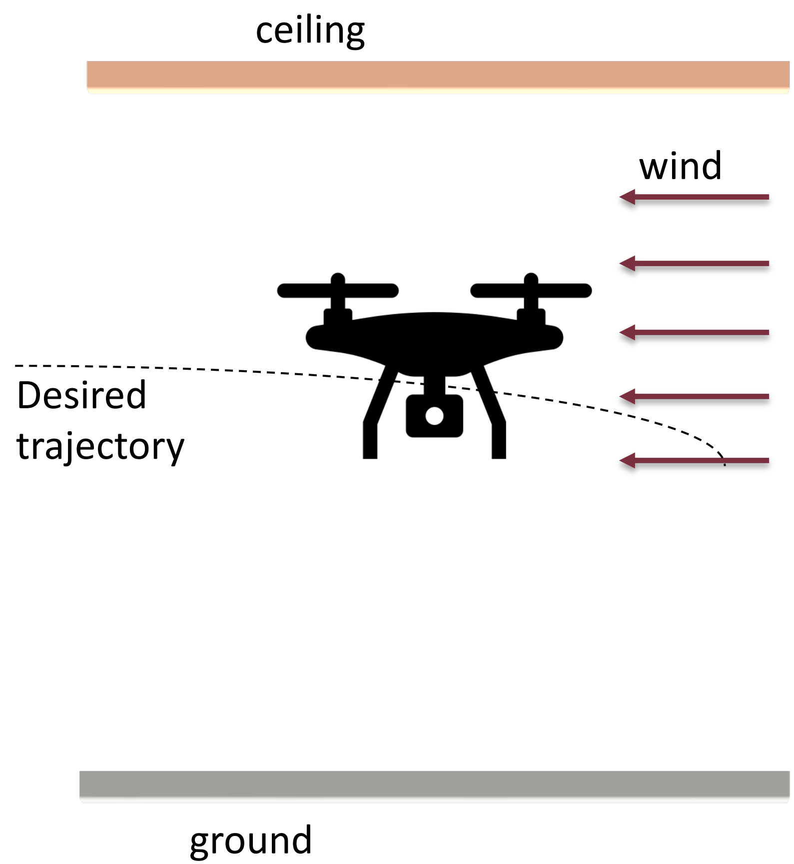
Today

Consider a drone in hover.

We want it to:

1. Track a desired trajectory, ☑
2. While not crashing into the ceiling or the ground, ☑
3. And account for disturbances.



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What kind of disturbances?

1. Gaussian distribution
2. Bounded uncertainty
3. Mean and variance
4. Discrete distribution

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1. Gaussian distribution
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We will only consider additive uncertainty. Consider an LTI system given by,

$$x(k+1) = Ax(k) + Bu(k) + Dw(k)$$

where, $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, w \in \mathbb{R}^{n_w}, A \in \mathbb{R}^{n_x \times n_x}, B \in \mathbb{R}^{n_x \times n_u}$, $D \in \mathbb{R}^{n_x \times n_w}$.

Assume that all the uncertainties are independent and identically distributed (i.i.d.)

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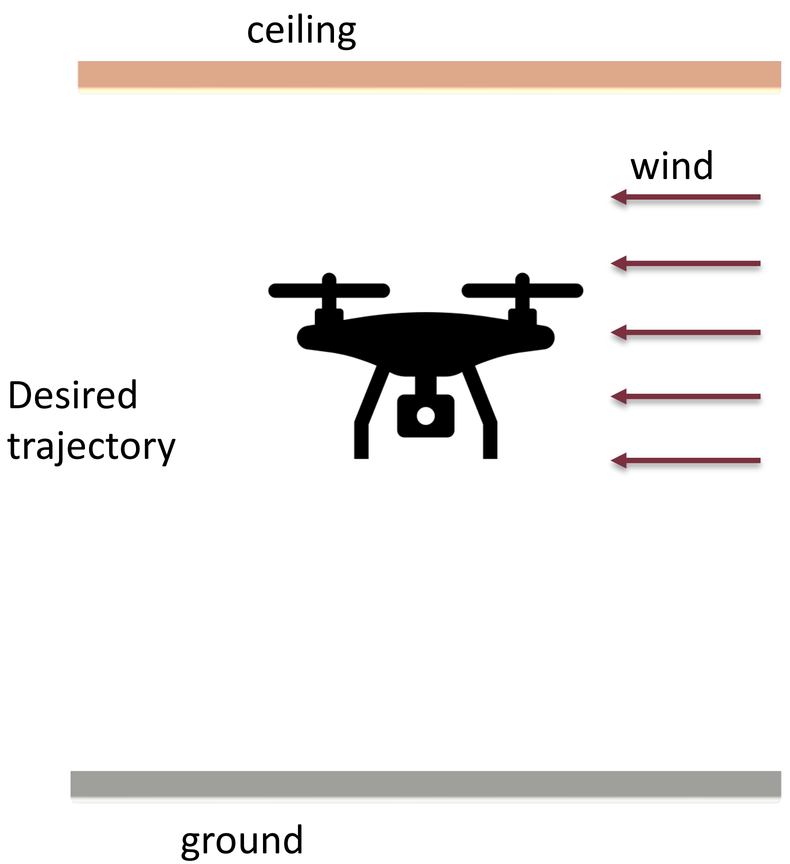
What kind of disturbances?

1. **Gaussian distribution**
2. Bounded uncertainty
3. Mean and variance
4. Discrete distribution



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Gaussian Disturbance

Assume that the i.i.d. disturbances have a Gaussian distribution with,

$$\mathbb{E}[w(k)] = \mathbf{0}_{n_w}, \quad \mathbb{E}\Big[\big(w(k) - \mathbb{E}[w(k)]\big)^T \big(w(k) - \mathbb{E}[w(k)]\big)\Big] = \Sigma$$

i.e., $w(k) \sim \mathcal{N}(\mathbf{0}_{n_w}, \Sigma)$.

The MPC optimization is given by,

$$\begin{aligned}
J_t^*(x(t)) = \min_{U_t} \quad & x_{t+N|t}^T P x_{t+N|t} + \sum_{k=t}^{t+N-1} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t} \right) \\
\text{s.t.} \quad & x_{k+1|t} = A x_{k|t} + B u_{k|t} + D w_{k|t}, \quad \forall k \in \{t, t+1, \dots t+N-1\} \\
& x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \qquad \qquad \qquad \qquad \forall k \in \{t, t+1, \dots t+N-1\} \\
& x_{t|t} = x(t)
\end{aligned}$$

How do we solve this?

Gaussian Disturbance

Assume that the i.i.d. disturbances have a Gaussian distribution.

$$
\begin{aligned}
J_t^*(x(t)) = \min_{U_t} \quad & x_{t+N|t}^T P x_{t+N|t} + \sum_{k=t}^{t+N-1} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t}\right) \\
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& x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \quad \forall k \in \{t, t+1, \ldots t+N-1\} \\
& x_{t|t} = x(t)
\end{aligned}
$$

We cannot solve the above problem in its current form. We need to pose it as a **deterministic** optimization.

A good deterministic measure of cost when the state is uncertain is its expected value.

Recall: Batch Approach without disturbance

In the absence of disturbances, we rewrite dynamics constraints (equality constraints) in terms of the initial condition and the control input as,

$$\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{X_0} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{\mathcal{S}_x} x(0) + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}}_{\mathcal{S}_u} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{U_0}$$

Hence, all the dynamics constraints can be written in batch form as,

$$X_0 = \mathcal{S}_x x(0) + \mathcal{S}_u U_0$$

Also let, $\bar{Q} = \text{blkdiag}(\underbrace{Q, Q, \dots, Q}_{\text{N times}}, P)$ and $\bar{R} = \text{blkdiag}(\underbrace{R, R, \dots, R}_{\text{N times}})$

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Gaussian disturbance: Cost Reformulation

Similarly, we rewrite the dynamics constraints (equality constraints) in terms of the initial condition, the control input, and the disturbances as,

$$\underbrace{\begin{bmatrix} x_{0|0} \\ x_{1|0} \\ x_{2|0} \\ \vdots \\ x_{N|0} \end{bmatrix}}_{X_0} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{\mathcal{S}_x} x_0 + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}}_{\mathcal{S}_u} \underbrace{\begin{bmatrix} u_{0|0} \\ u_{1|0} \\ u_{2|0} \\ \vdots \\ u_{N-1|0} \end{bmatrix}}_{U_0} + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ D & 0 & \dots & 0 \\ AD & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}D & A^{N-2}D & \dots & D \end{bmatrix}}_{\mathcal{S}_w} \underbrace{\begin{bmatrix} w_{0|0} \\ w_{1|0} \\ w_{2|0} \\ \vdots \\ w_{N-1|0} \end{bmatrix}}_{W_0}$$

Hence, all the dynamics constraints can be written in batch form as,

$$X_0 = \mathcal{S}_x x_0 + \mathcal{S}_u U_0 + \mathcal{S}_w W_0, \qquad W_0 \sim \mathcal{N}(\mathbf{0}, \bar{\Sigma})$$

$$\bar{\Sigma} = \mathrm{blkdiag}(\Sigma, \dots, \Sigma)$$

and the expected cost is given by,

$$\begin{aligned} J(x(0), U_0) &= \mathbb{E}\left[X_0^T \bar{Q} X_0 + U_0^T \bar{R} U_0\right] \\ &= (\mathcal{S}_x x(0) + \mathcal{S}_u U_0)^T \bar{Q} (\mathcal{S}_x x(0) + \mathcal{S}_u U_0) \\ &\quad + U_0^T \bar{R} U_0 + \mathrm{Tr}(\bar{\Sigma} \mathcal{S}_w^T \bar{Q} \mathcal{S}_w) \end{aligned}$$

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Gaussian disturbance: Constraint Reformulation

Assume that the i.i.d. disturbance is Gaussian, $w(k) \sim \mathcal{N}(\mathbf{0}_{n_w}, \Sigma)$,

$$
\begin{aligned}
J_t^*(x(t)) = \min_{U_t} \quad & x_{t+N|t}^T P x_{t+N|t} + \sum_{k=t}^{t+N-1} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t} \right) \\
\text{s.t.} \quad & x_{k+1|t} = A x_{k|t} + B u_{k|t} + D w_{k|t}, \quad \forall k \in \{t, t+1, \ldots t+N-1\} \\
& x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \quad \forall k \in \{t, t+1, \ldots t+N-1\} \\
& x_{t|t} = x(t)
\end{aligned}
$$

Now we can replace the random cost function with its expectation.

What about the constraints? What is a good deterministic constraint to implement here?

Perhaps expectation again?

Chance constraints

For a random variable w, a chance constraint satisfies the inequality

$$\mathrm{Prob}(f(x,w) > 0) \leq \epsilon$$

This means a constraints of the form $f(x,w) \leq 0$ holds with at least $1 - \epsilon$ confidence.

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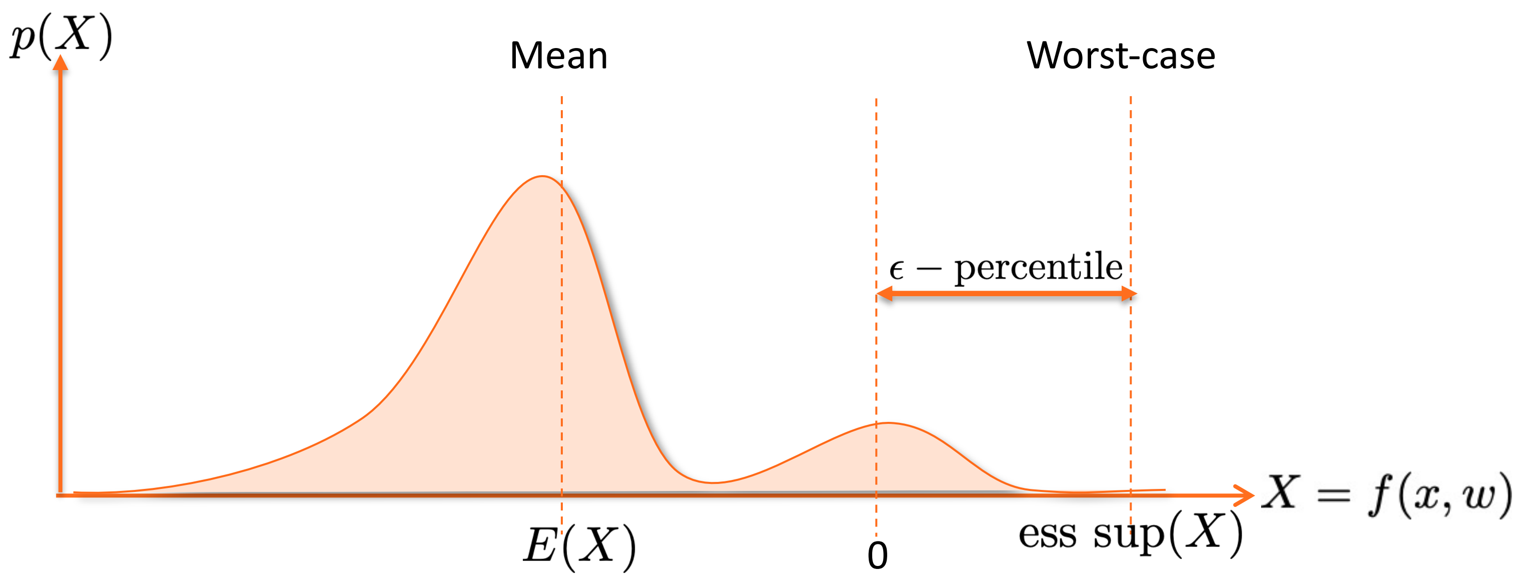

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Chance Constraints for Gaussian disturbances

Chance Constraints for Linear Inequality:

For a normal random variable $w \sim \mathcal{N}(0, \Sigma)$, we want to satisfy

$$\mathrm{Prob}(a^T w + x > 0) \leq \epsilon, \quad a, w \in \mathbb{R}n_w,\ x \in \mathbb{R}$$

It follows that $a^T w + x \sim \mathcal{N}(x, a^T \Sigma a)$

Hence,
$$\begin{aligned}\mathrm{Prob}(a^T w + x > 0) &= 1 - \mathrm{Prob}(a^T w + x \leq 0) \\ &= 1 - \underbrace{\Phi\left(\frac{-x}{\sqrt{a^t \Sigma a}}\right)}_{\text{cdf of } \mathcal{N}(0,1)}\end{aligned}$$

Or we can equivalently write the chance constraint as

$$x + \|\Sigma^{1/2} a\|_2 \Phi^{-1}(1 - \epsilon) \leq 0$$

Gaussian disturbance: Constraint Reformulation

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& x_{t|t} = x(t)
\end{aligned}
$$

We wrote the equality constraints as,

$$X_0 = \mathcal{S}_x x_0 + \mathcal{S}_u U_0 + \mathcal{S}_w W_0, \qquad W_0 \sim \mathcal{N}(\mathbf{0}, \bar{\Sigma})$$

$\bar{\Sigma} \to \text{blkdiag}(\Sigma, \ldots, \Sigma)$

Equivalently, the state inequality constraints are given by,

$$\bar{F}_x X_0 = \bar{F}_x \left(\mathcal{S}_x x_0 + \mathcal{S}_u U_0 + \mathcal{S}_w W_0\right) \leq \bar{b}_x$$

$\bar{F}_x \to \text{blkdiag}(F_x, \ldots, F_x)$

$\bar{b}_x \to [b_x, \ldots, b_x]^T$

Gaussian disturbance: MPC Reformulation

Assume that the i.i.d. disturbance is Gaussian, $w(k) \sim \mathcal{N}(\mathbf{0}_{n_w}, \Sigma)$.
The MPC cost is expressed as an expectation and the constraints are expressed as chance constraints, i.e.,

$$
\begin{aligned}
J_t^*(x(t)) = \min_{U_t} \quad & \mathbb{E}\Big[x_{t+N|t}^T P x_{t+N|t} + \sum_{k=t}^{t+N-1} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t}\right)\Big] && \text{(Stochastic MPC)} \\
\text{s.t.} \quad & x_{k+1|t} = A x_{k|t} + B u_{k|t} + D w_{k|t}, && \forall k \in \{t, t+1, \dots t+N-1\} \\
& \text{Prob}(x_{k|t} \notin \mathcal{X}) \leq \epsilon, \; u_{k|t} \in \mathcal{U}, && \forall k \in \{t, t+1, \dots t+N-1\} \\
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& x_{t|t} = x(t)
\end{aligned}
$$

The above stochastic optimization problem is reformulated as

$$
\begin{aligned}
J_t^*(x(t)) = \min_{U_t} \quad & (\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t)^T \bar{Q} (\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t) + U_t^T \bar{R} U_t + \underbrace{\text{Tr}(\bar{\Sigma} \mathcal{S}_w^T \bar{Q} \mathcal{S}_w)}_{\text{Constant}} \\
\text{s.t.} \quad & \bar{F}_x \big(\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t\big) + \Phi^{-1}(1-\epsilon) \underbrace{\|\bar{F}_x S_w \bar{\Sigma}^{1/2}\|_2}_{\text{Vector-norm of each row of the matrix}} \leq \bar{b}_x, \quad \forall k \in \{t, t+1, \dots t+N-1\} \\
& u_{k|t} \in \mathcal{U}, \quad \forall k \in \{t, t+1, \dots t+N-1\} \\
& x_{t|t} = x(t)
\end{aligned}
$$

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What kind of disturbances?

1. Gaussian distribution
2. **Bounded uncertainty**
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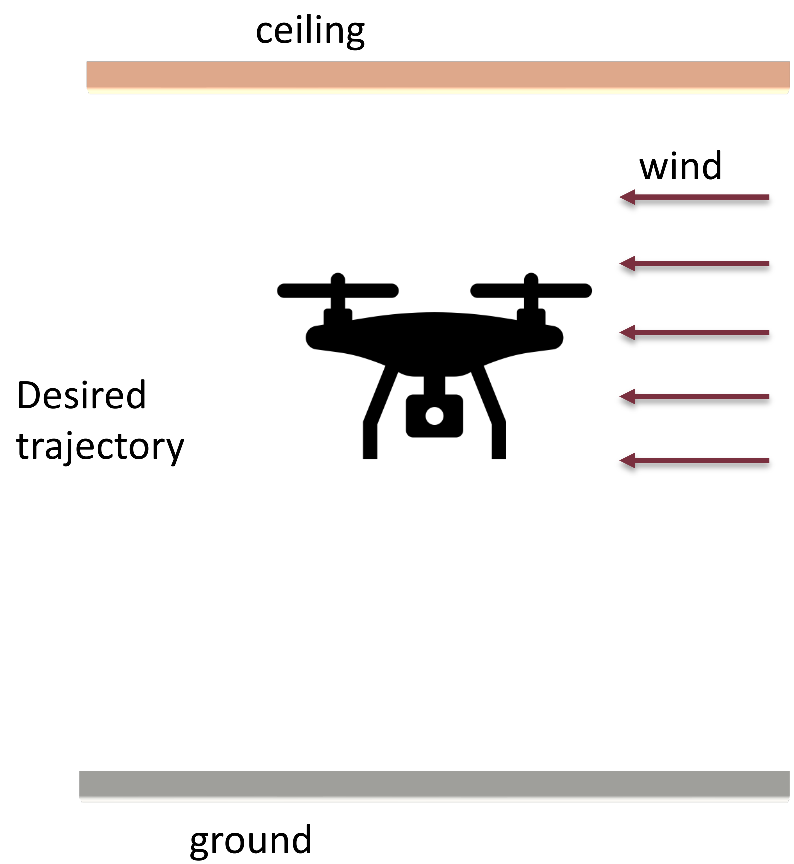

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Bounded disturbance

Assume that the i.i.d. disturbances are bounded and lie in a convex polytope, i.e., $w(k) \in \mathcal{W}$ where,

$$\mathcal{W} = \{w \in \mathbb{R}^{n_w} : F_w w \leq b_w\}$$

Since every disturbance from this set is equally likely, we would like to **account for all possible disturbances.**

How do we frame the MPC problem so that it can do this?

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Bounded disturbance

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$$\mathcal{W} = \{w \in \mathbb{R}^{n_w} : F_w w \leq b_w\}$$

Since every disturbance from this set is equally likely, we would like to **account for all possible disturbances.**

$$\begin{aligned}
J_t^*(x(t)) = \min_{U_t} \quad & \max_{W_t \in \mathcal{W}} \left[x_{t+N|t}^T P x_{t+N|t} + \sum_{k=t}^{t+N-1} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t} \right) \right] \\
\text{s.t.} \quad & x_{k+1|t} = A x_{k|t} + B u_{k|t} + D w_{k|t}, \quad \forall k \in \{t, t+1, \ldots t+N-1\} \\
& x_{k|t} \in \mathcal{X}, \qquad \forall w_{k|t} \in \mathcal{W}, \forall k \in \{t, t+1, \ldots t+N-1\} \\
& u_{k|t} \in \mathcal{U}, \qquad \forall k \in \{t, t+1, \ldots t+N-1\} \\
& x_{t|t} = x(t)
\end{aligned}$$

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Maxima over convex polytopes

Lemma:

Let $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_g}$ be a function of (x, w) convex in w for each x. Assume that $w \in \mathcal{W}$ where $\mathcal{W}$ is a polytope with vertices $\{\bar{w}_i\}_{i=1}^{n_\mathcal{W}}$. Then the constraint,

$$g(x, w) \leq 0 \quad \forall w \in \mathcal{W}$$

is satisfied iff

$$g(x, \bar{w}_i) \leq 0, \forall i \in \{1, \ldots, n_\mathcal{W}\}$$

This means that the maximum of a function convex in w, always occurs on the vertices of the polytope we are maximizing over.

See Lemma 10.1 in the textbook

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Bounded Disturbance: Constraint Reformulation

We've seen that our state constraints can be expressed as,

$$\bar{F}_x X_t = \bar{F}_x\big(\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t + \mathcal{S}_w W_t\big) \leq \bar{b}_x$$

Here, $W_t \in \mathcal{W} \times \mathcal{W} \times \ldots \times \mathcal{W}$
We can enumerate all the vertices of the set given by $\mathcal{W} \times \mathcal{W} \times \ldots \times \mathcal{W}$ as $\{\bar{W}_{t,i}\}_{i=1}^{n_v}$, where $n_v = n_{\mathcal{W}\times\ldots\times\mathcal{W}}$.

Now the state constraint can be reformulated as a set of constraints,

$$\bar{F}_x\big(\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t + \mathcal{S}_w \bar{W}_{t,i}\big) \leq \bar{b}_x,\ \forall i \in \{1,\ldots,n_v\}$$

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Bounded Disturbance: Cost Reformulation

We've seen that our cost can be expressed as,

$$J(x(t), U_t, W_t) = (\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t + \mathcal{S}_w W_t)^T \bar{Q} (\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t + \mathcal{S}_w W_t) + U_t^T \bar{R} U_t$$

The worst-case cost over all possible disturbances is

$$\begin{aligned} J_{\text{robust}}(x(t), U_t) &= \max_{W_t \in \mathcal{W} \times \ldots \times \mathcal{W}} J(x(t), U_t, W_t) \\ &= \max_{W_t \in \{\bar{W}_{t,i}\}_{i=1}^{n_v}} J(x(t), U_t, W_t) \end{aligned}$$

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Bounded Disturbance: MPC Reformulation

Given:

$$J(x(t), U_t, W_t) = (\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t + \mathcal{S}_w W_t)^T \bar{Q} (\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t + \mathcal{S}_w W_t) + U_t^T \bar{R} U_t$$

The following MPC optimization

$$J_t^*(x(t)) = \min_{U_t} \quad \max_{W_t \in \mathcal{W}} \left[x_{t+N|t}^T P x_{t+N|t} + \sum_{k=t}^{t+N-1} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t} \right) \right]$$

$$\begin{aligned} \text{s.t.} \quad & x_{k+1|t} = A x_{k|t} + B u_{k|t} + D w_{k|t}, && \forall k \in \{t, t+1, \dots t+N-1\} \\ & x_{k|t} \in \mathcal{X}, && \forall w_{k|t} \in \mathcal{W}, \forall k \in \{t, t+1, \dots t+N-1\} \\ & u_{k|t} \in \mathcal{U}, && \forall k \in \{t, t+1, \dots t+N-1\} \\ & x_{t|t} = x(t) \end{aligned}$$

can be reformulated as a **quadratically-constrained** optimization,

$$J_t^*(x(t)) = \min_{U_t} \quad m$$

$$\begin{aligned} \text{s.t.} \quad & J(x_{t|t}, U_t, \bar{W}_{t,i}) \leq m, \\ & \bar{F}_x \left(\mathcal{S}_x x_{t|t} + \mathcal{S}_u U_t + \mathcal{S}_w \bar{W}_{t,i} \right) \leq \bar{b}_x, \\ & u_{k|t} \in \mathcal{U}, \qquad \forall i \in \{1, \dots, n_v\}, \forall k \in \{t, t+1, \dots t+N-1\} \\ & x_{t|t} = x(t) \end{aligned}$$

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Summary

We considered two types of uncertainty descriptions,

- Gaussian disturbances (unbounded)
- Polytopic uncertainty (bounded)

We saw two different ways of reformulating the MPC:

- In the Gaussian case, we looked at the **expectation** of the cost and bounded the **probability of constraint violation**.

- In the case of polytopic uncertainty, we treated the uncertainty as adversarial. We planned for the **worst-case**, in both the cost and the constraints.

- **Next Lecture:** Was the expected value the right cost to consider? How about the worst-case cost? How do we reformulate our constraints if the distribution is not Gaussian or polytopic?